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Quantum Algebra of the Particle Moving on the q-Deformed Mass-Hyperboloid[†]

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Abstract

I introduce a reality structure on the Heisenberg double of $\mathrm{Fun}_q(SL(N,\mathrm{C}))$ for q phase, which for $N=2$ can be interpreted as the quantum phase space of the particle on the q-deformed mass-hyperboloid. This construction is closely related to the q-deformation of the symmetric top. Finally, I conjecture that the above real form describes zero modes of certain non-compact WZNZ-models.

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Disclaimer

1 Introduction

Monodromy matrices representing the braid group [1], appearing in the WZNZ-model, suggested that hidden quantum groups exist in these theories. Various approaches were used in an attempt to elucidate the origin of these hidden quantum groups. In [2, 3, 4, 5] using a Minkowski space-time lattice regularization, it was shown by explicit construction that the monodromies of the chiral components of the WZNW-model with Lie group G and the local field satisfy the commutation relations of the q-deformed cotangent bundle T^*G_q.

However an apparent contradiction existed [3, 5], since the deformation parameter in the WZNW-model must be root of unity $q = \exp(i\pi/k + h)$, where k is the level of the affine-Lie algebra and h is the dual Coxeter number, and this is incompatible with the compact form of the quantum group.

A solution to this problem was proposed in [6]. The main idea is to drop the strong requirement that the reality structure be compatible with quantum group comultipication and only impose this requirement in the classical limit. Then a reality structure can be introduced, but not on the quantum group itself, but rather on the quantum cotangent bundle.

However once the requirement of the compatibility of the reality structure with the comultiplication is dropped, one can introduce more than one reality structure. In this paper I will introduce one such reality structure inspired by a particular type of non-compact WZNW-model. See for example [7] for a list of various circumstances under which this non-compact form occurs and also [8] where the non-compact form of appears as the Euclidean section of the model. These WZNW-models have the important property that the local field has the chiral decomposition $g = hh^\dagger$ where h is the chiral field valued in G. Thus g is a Hermitian positive defined matrix of unit determinant. I will show that

$$g^\dagger = g$$

is compatible with the algebra T^*G_q and extend the above anti-involution to the whole algebra. I emphasize that the reality structure introduced here is similar to the one discussed in [6] and is not related to the standard non-compact reality structure appearing in quantum groups for q phase, and which is compatible with comultiplication.

For simplicity here I will not apply the reality structure directly in the WZNW-model, leaving this for a forthcoming paper, and instead I will just use it for the toy

model of [5, 6], which essentially contains all the relevant degrees of freedom. These degrees of freedom are described by the same algebra as in the compact case but with a different reality structure.

In Section 2, I give a short review of the quantum algebra T^*G_q. I discuss the commutation relations for operators generating both left and right translations, since both forms are necessary to define or to check the involutions presented in the next sections. Section 3 briefly covers the reality structure of [6]. In section 4, I present the main result of the paper, a reality structure corresponding to a generalized mass-hyperboloid configuration space and its associated q-deformed phase space. In Section 5, I consider the simple quantum mechanical system of [6] and show its compatibility with the $*$-structure introduced in the previous section. In the last Section I present some evidence for the relevance of this reality structure to the non-compact WZNW-model.

2 Review of the Algebra on T^*G_q

In this section I present a brief review of the defining relations of the q-deformed cotangent bundle [5] also known as the Heisenberg double or as the smash product [9, 10]. The main purpose of this section is to fix the notation. I will follow closely the presentation in [6] where a more detailed exposition can be found.

Let G be the Lie group $SL(N,\mathrm{C})$, and sometimes for simplicity I will take $G = SL(N,2)$. Most of the content of the paper can be easily extended to arbitrary classical groups. Now consider the quantum R_+ matrix associated to the Lie group G. This is a matrix depending on a parameter q and acting in the tensor product of two fundamental representations. For example the R_+ of $SL(2,\mathrm{C})$ is the following 4×4 matrix

$$R_+ = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

where $\lambda = q - q^{-1}$. It is convenient to also use the R_- matrix defined as

$$R_- = PR_+^{-1}P \tag{1}$$

where P is the permutation operator in the tensor space of the two fundamental representations

$$P(a\otimes b) = b\otimes a.$$

Next I will define the quantum algebra T^*G_q, the quantum deformation of the cotangent bundle. Let g and $\Omega_\pm$ be matrices acting in the fundamental representation of G. The $\Omega_\pm$ matrices are upper and lower triangular matrices. In addition the diagonal elements of Ω_+ equal those of Ω_-^{-1}. T^*G_q is the algebra generated by g and $\Omega_\pm$ and satisfying the following set of relations divided for convenience into three groups

$$R_\pm g^1 g^2 = g^2 g^1 R_\pm \tag{2}$$

$$\begin{aligned} R_\pm \Omega_+^1 \Omega_+^2 &= \Omega_+^2 \Omega_+^1 R_\pm \\ R_\pm \Omega_-^1 \Omega_-^2 &= \Omega_-^2 \Omega_-^1 R_\pm \\ R_+ \Omega_-^1 \Omega_-^2 &= \Omega_-^2 \Omega_+^1 R_+ \\ R_- \Omega_-^1 \Omega_+^2 &= \Omega_+^2 \Omega_-^1 R_- \end{aligned} \tag{3}$$

$$\begin{aligned} R_+ \Omega_+^1 g^2 &= g^2 \Omega_+^1 \\ R_- \Omega_-^1 g^2 &= g^2 \Omega_-^1 . \end{aligned} \tag{4}$$

All the above relations are operator matrices acting in the tensor product of two fundamentals, and the superscript indicates on which factor the respective matrix acts. The R matrices without any superscript act in both spaces. One can show that the quantum determinant of the matrices g and $\Omega_\pm$ is central and can be set equal to one

$$\det_q(g) = \det_q(\Omega_\pm) = 1.$$

For the $SL(N, \mathrm{C})$ groups these are all the relations, while for the other classical groups additional relations, for example orthogonality relations, have to be imposed. Noto also that, unlike (2)(3), the relation (4) is not homogeneous in $R_\pm$ thus the normalization of $R_\pm$ is important.

The above relations are not independent. For example the R_- relations can be obtained from the R_+ relations using (1) and

$$X^2 = P X^1 P. \tag{5}$$

The subalgebra generated by the matrix elements of g with relations (2) is in fact a Hopf algebra denoted $\mathrm{Fun}_q(G)$ and represents a deformation of the Hopf algebra of function on the G Lie group [11]. Also, the subalgebra generated by $\Omega_\pm$ with

relations (3) is a quasitriangular Hopf algebra called the quantum universal enveloping algebra [12, 13, 11], and is denoted $U_q(\mathbf{g})$ where the $\mathbf{g}$ in the brackets is the Lie algebra of the Lie group G. For example the coproduct of $\mathrm{Fun}_q(G)$ on the matrix elements of g is given by

$$\triangle(g) = g\dot{\otimes}g, \tag{6}$$

where the dot means multiplication in matrix space. Similarly the coproduct in $U_q(\mathbf{g})$ on the matrix elements $\Omega_\pm$ reads

$$\triangle(\Omega_\pm) = \Omega_\pm\dot{\otimes}\Omega_\pm. \tag{7}$$

On the other hand T^*G_q is not a Hopf algebra. We emphasize this, since there is a related algebra, the Drinfeld double, which has the same generators but different mixed relations and is a Hopf algebra.

The mixed relations (4) describe how to combine the above subalgebras into the larger algebra T^*G_q. They appear as commutation relations in [5, 9, 10] but in an abstract form as the pairing of dual Hopf algebras they were already present in [11].

One can relate the $\Omega_\pm$ with the more traditional Drinfeld-Jimbo generators. For example for the $SL(2, \mathrm{C})$ group we can write the matrix elements of $\Omega_\pm$ as [11]

$$\Omega_+ = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \; \Omega_- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_- & q^{-H/2} \end{pmatrix}. \tag{8}$$

Usinq the R_+ matrix above it can be shown by direct computations that the generators $H, X_\pm$ satisfy the Jimbo-Drinfeld relations [12, 13]

$$[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{9}$$

defining the universal enveloping algebra $\mathcal{U}_q(sl(2, \mathrm{C}))$. Similar relations also exist for higher rank groups [11] and can be thought of as connecting the Cartan-Weyl and Chevalley bases.

It is also convenient to combine $\Omega_\pm$ into a single matrix [14]

$$\Omega = \Omega_+\Omega_-^{-1}. \tag{10}$$

In terms of these generators all the relations (3) and (4) collapse to

$$\begin{aligned} \Omega^1 R_-^{-1}\Omega^2 R_- &= R_+^{-1}\Omega^2 R_+\Omega^1 \\ R_- g^1\Omega^2 &= \Omega^2 R_+ g^1. \end{aligned} \tag{11}$$

These forms of the commutation relations are especially useful when we deal with the commutation relations only, but the coproduct of Ω cannot in general be given in an explicit form.

The commutation relations (2)(11) are exactly those satified by the local field and the monodromy of the left (or right) chiral component of the affine current [2, 3, 4].

Following [6] we also introduce an equivalent description of the quantum algebra using operators generating right translations. First let

$$\Sigma = g^{-1}\Omega g,$$

and then introduce a triangular decomposition of Σ into $\Sigma_\pm$

$$\Sigma = \Sigma_+\Sigma_-^{-1} \tag{12}$$

similar to the decomposition of Ω into $\Omega_\pm$. One can check that the matrix elements of Ω and Σ commute. To make the picture more symmetric also introduce a new matrix h by

$$h = \Sigma_\pm^{-1} g^{-1}\Omega_\pm. \tag{13}$$

Now we can use either pair (g, Ω) or (h, Σ) to describe the algebraT$^*G_q$.

The defining relations satisfied by h and Σ are [6]

$$\begin{array}{rcl}
R_\pm h^1 h^2 & = & h^2 h^1 R_\pm \\
\Sigma_+^1\Sigma_+^2 R_\pm & = & R_\pm \Sigma_+^2\Sigma_+^1 \\
\Sigma_-^1\Sigma_-^2 R_\pm & = & R_\pm \Sigma_-^2\Sigma_-^1 \\
\Sigma_-^1\Sigma_+^2 R_+ & = & R_+ \Sigma_-^2\Sigma_-^1 \\
\Sigma_+^1\Sigma_-^2 R_- & = & R_- \Sigma_-^2\Sigma_+^1 \\
h^1\Sigma_+^2 & = & \Sigma_+^2 R_- h^1 \\
h^1\Sigma_-^2 & = & \Sigma_-^2 R_+ h^1.
\end{array} \tag{14}$$

One can check directly the consistency of (14) with the original relations.

3 Real Form for the q-Deformed Symmetric Top

For a large number of applications the variable q is a phase. In this case the $R_\pm$ matrices satisfy

$$R_+^\dagger = R_-. \tag{15}$$

If we require a reality structure for g compatible with the Hopf algebra structure i.e.

$$\triangle \circ * = (* \otimes *) \circ \triangle$$

and use (15) we obtain a non-compact quantum group. For example if $G = SL(N, \mathrm{C})$ we obtain $\mathrm{Fun}_q(SL(N, \mathrm{R}))$.

However sometimes in the same application we are interested in the compact form of the group. This apparent contradiction can be resolved [6] by dropping the above requirement for a Hopf $*$-structure. Instead one defines an anti-involution on the larger algebra $\mathrm{T}^* G_q$

$$\Omega_\pm^\dagger = \Omega_\mp \tag{16}$$

$$g^\dagger = h. \tag{17}$$

It is straightforward [6] to check the compatibility of this anti-involution with the quantum algebra (2)(3)(4)(14). Note that (16) does not define a Hopf $*$-structure on $U_q(\mathfrak{g})$, and (17) does not close on $\mathrm{Fun}_q(G)$ since the definition of h includes generators of $U_q(\mathfrak{g})$. In the classical limit (17) reduces to $g^\dagger = g^{-1}$ and (16) becomes compatible with the coproduct. This is due to the fact that the coproduct is cocommutative at$q = 1$.

4 Real Form for the q-Deformed Hyperboloid

This section contains the main result of the paper, an anti-involution on the deformed cotangent bundle when q is a phase. Like the anti-involution of the previous section, it does not originate from a Hopf $*$-structure on one of the Hopf subalgebras. The defining relations of the anti-involution are

$$g^\dagger = g \tag{18}$$

$$\Omega_\pm^\dagger = \Sigma_\mp^{-1}. \tag{19}$$

Alternatively the second relation can be written as

$$\Omega^\dagger = \Sigma = g^{-1} \Omega g. \tag{20}$$

It is quite obvious that (18) is not compatible with the coproduct, i.e. g should not be considered a "group element". I will not give a complete proof of the consistency

of the anti-involution with the algebra relations (2)(3)(4). Instead I will just give a sample computation leaving the rest for the interested reader.

Applying the involution on the R_+ relation (2) and using (15) we have

$$(g^2)^\dagger (g^1)^\dagger R_- = R_-(g^1)^\dagger (g^2)^\dagger.$$

Moving the R_- matrices to the other side and using (1) we obtain

$$R_+(g^1)^\dagger (g^2)^\dagger = (g^2)^\dagger (g^1)^\dagger R_+,$$

thus it is consistent with the algebra relations (2) to impose $g^\dagger = g$.

As another example, take the hermitian conjugate of the following relation

$$R_+\Omega^1_+\Omega^2_+ = \Omega^2_+\Omega^1_+R_+. \tag{21}$$

Using (19) we obtain

$$(\Sigma^2_-)^{-1}(\Sigma^1_-)^{-1}R_- = R_-(\Sigma^1_-)^{-1}(\Sigma^2_-)^{-1}$$

which can be rewritten after multiplication by some inverse matrices as

$$R_-\Sigma^2_-\Sigma^1_- = \Sigma^1_-\Sigma^2_-R_-.$$

This is just one of the equations in (14).

Similarly applying the above involution on the first relation in (4) we obtain

$$g^2(\Sigma^1_-)^{-1}R_- = (\Sigma^1_-)^{-1}g^2$$

$$\Sigma^1_-g^2 = g^2R_-^{-1}\Sigma^1_-.$$

This is equivalent using (1) and (5) to

$$\Sigma^2_-g^1 = g^1R_+\Sigma^2_-,$$

and after eliminating g using (13) we get

$$\Sigma^2_-\Omega^1_-(h^1)^{-1}(\Sigma^1_-)^{-1} = \Omega^1_-(h^1)^{-1}(\Sigma^1_-)^{-1}R_+\Sigma^2_-.$$

Furthermore using (14) to commute the Σ matrices we have

$$\Sigma^2_-\Omega^1_-(h^1)^{-1} = \Omega^1_-(h^1)^{-1}\Sigma^2_-R_+$$

and since Ω and Σ commute with each other we finally obtain

$$h^1 \Sigma_-^2 = \Sigma_-^2 R_+ h^1$$

which is again one of the relations in (14). All the other relations can be checked in a similar fashion.

Finally I will explain the terminology used in the title of this section. Consider first for simplicity the $SL(2,\mathrm{C})$ case. In the undeformed case a 2×2 hermitian matrix of unit determinant defines the unit mass hyperboloid in Minkowski space. For simplicity I will only consider one connected component of the manifold, for example the future mass hyperboloid. For a general group G this can be achieved by restricting to positive definite matrices. In the deformed case we consider Hermitians g matrices of unit quantum determinant.

5 Quantum Mechanics on the q-Deformed Hyperboloid

In [5] Alekseev and Faddeev showed that the $\mathrm{T}^* G_q$ quantum algebra is a q-deformation of the algebra of functions on the cotangent bundle of the Lie group G. In [6] they considered the following simple Lagrangian written in first order formalism

$$\mathcal{L} = \mathrm{Tr}(\omega \dot{g} g^{-1} - \frac{1}{2}\omega^2). \tag{22}$$

Here G is considered without specifying its real form. The Lagrangian has a chiral symmetry $G \times G$

$$g \rightarrow u g v^{-1}, \quad \omega \rightarrow u \omega v^{-1}, \quad u, v \in G.$$

The second order form of the Lagrangian has the form of a non-linear sigma model in $(0,1)$ dimensions

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}(\dot{g} g^{-1} \ \dot{g} g^{-1}). \tag{23}$$

The equations of motion

$$\dot{g} = \omega g, \quad \dot{\omega} = 0$$

can be integrated to give the time evolution

$$\begin{aligned} \omega(t) &= \omega(0) \\ g(t) &= \exp(\omega t)\ g(0). \end{aligned}$$

The real form corresponding to the compact group discussed in [6] is

$$g^\dagger = g^{-1}, \quad \omega^\dagger = -\omega. \tag{24}$$

For $G = SL(2,\mathrm{C})$, g becomes unitary and the Lagrangian (22) describes the classical dynamics of the symmetric top. Equivalently, it describes the motion on a constant curvature S^3. This can be seen using the chiral symmetry (5) of the Lagrangian, which under the conditions (24) is restricted to the $SU(2) \times SU(2) \sim SO(4)$ subgroup, or by direct computation of the metric in the kinetic term of (23).

Instead, we consider the following reality structure

$$g^\dagger = g, \quad \omega^\dagger = g^{-1}\omega g \tag{25}$$

which, following from the discussion at the end of the previous, section defines the phase space of a particle moving on the mass-hyperboloid. The reality structure (25) requires $u^\dagger = v^{-1}$ thus restricting the chiral symmetry of the Lagrangian to one independent $SL(2,\mathrm{C})$ subgroup which is simply the Lorentz group that leaves the mass hyperboloid invariant. The metric on the hyperboloid is just the induced metric from Minkowski space, and again this can be obtained by direct computation or using the above invariance under the Lorentz group.

One can check that the equations of motion preserve both reality structures (24) and (25). What we learn from this simple example is that one can find rather different physical systems that will have the same Poisson brackets and thus quantum algebras if their respective Lagrangians have the same form, differing only through their reality structures.

In [6] a q-deformation of the above system was introduced. The model has a discrete time dynamics, with the time labelled by an integer n. The following evolution equations

$$\begin{aligned} \Omega(n) &= \Omega(0) \\ g(n) &= \Omega^n g(0) \end{aligned} \tag{26}$$

were shown in [6] to preserve the quantum algebra (2)(3)(4) and in addition, the reality structure discussed in Section 3.

I will now show that they also preserve the reality structure introduced in Section 4. Assuming that for $n = 0$ the reality structure is given by (18) and (20)

$$g^\dagger(0) = g(0), \quad \Omega^\dagger(0) = g^{-1}(0)\ \Omega(0)\ g(0)$$

for arbitrary n we have

$$g^\dagger(n) = g^\dagger(0)(\Omega^\dagger(0))^n = g(0)(g^{-1}(0)\Omega())g(0))^n = \Omega^n(0)g(0) = g(n).$$

Similarly we have for $\Omega(n)$

$$\Omega^\dagger(n) = \Omega^\dagger(0) = g^{-1}(0)\Omega(0)g(0) = g^{-1}(n)\Omega(0)g(n) = g^{-1}(n)\Omega(n)g(n).$$

Thus the equations of motion (26) and the reality structure of the previous Section define the q-deformation of the dynamics of a particle on the unit mass hyperboloid.

6 Concluding Remarks

I conclude by briefly applying the reality structure to the lattice regularized WZNW-model and checking its compatibility with periodic boundary conditions. Using the notation in [2] let the lattice have N points, and denote the local fields by $g_i,\ i = 1\ldots N$. For periodic boundary conditions we identify i and $i+N$. Let M_L and M_R be the monodromies of the left and right affine currents. The algebra satisfied by (g, M_L, M_R) is exactly the algebra of T^*G_q for the generators (g, Ω, Σ). Here I used the remark of the previous Section that the compact and non-compact WZNW-models have the same algebra since their respective Lagrangians coincide. The monodromies can be used to relate the fields g_0 and g_N

$$g_N = M_L g_0 M_R^{-1}$$

If we require $g_0^\dagger = g_0,\quad M_L^\dagger = M_R$, which is just the reality structure of Section 4, we have

$$g_N^\dagger = (M_R^{-1})^\dagger g_0 M_L^\dagger = M_L^{-1} g_0 M_R = g_{-N} = g_N.$$

In the last step I used the lattice periodicity. Thus we see that the reality structure is compatible with periodic boundary conditions. A more detailed investigation of the implications of this reality structure for the WZNW-model will be presented in an upcoming paper.

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References

[1] A. Tsuchiya, Y. Kanie *Vertex Operators in the Conformal Field Theory on* P^1 *and Monodromy Representations of the Braid Group*, Letters in Math. Phys. 13 (1097) 303-312

[2] A. Yu. Alekseev, L.D. Faddeev, M. A. Semenov-Tian-Shansky, A. Volkov *The Unraveling of the Quantum Group Structure in the WZNW theory*, Preprint CERN-TH-5981/91, January1991

[3] A. Yu. Alekseev, L.D. Faddeev, M. A. Semenov-Tian-Shansky *Hidden Quantum Groups Inside Kac-Moody Algebra*, Commun. Math. Phys. 149 (1992) 335-345

[4] L.D. Faddeev, *From Integrable Models to Conformal Field Theory via Quantum Groups*, Integrable Systems, Quantum Groups, and Quantum Field Theory, L. A. Ibort, M. A. Rodríquez (eds.)

[5] A. Yu. Alekseev, L.D. Faddeev $(\mathrm{T}^*G)_t$*: A Toy Model for Conformal Field Theory*, Commun. Math. Phys. 141 (1991) 413-422

[6] A. Yu. Alekseev, L.D. Faddeev *An Involution and Dynamics for the q-Deformed Quantum Top*, Preprint hep-th/9406196, June 1994

[7] K. Gawędzki, *Non-Compact WZW Conformal Field Theories*, Preprint hep-th/9110076, October 1991

[8] C. Destri, H. J. De Vega *On The Connection Between The Principal Chiral Model and the Multiflavour Chiral Gross-Neveu Model*, Phys. Lett. B 201 (1988) 245-250´

[9] B. Zumino, *Introduction to the Differential Geometry of Quantum Groups*, K. Schmüdgen (Ed.), Math. Phys. X, Proc. X-th IAMP Conf. Leipzig (1990), Springer-Verlag (1991)

[10] B. Zumino, *Differential Calculus on Quantum Spaces and Quantum Groups*, XIX ICGTMP, M. O., M. S. and J. M. G. (Ed.), CIEMAT/RSEF, Madrid, vol. 1 (1993) 4

[11] L.D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan *Quantization of Lie Groups and Lie Algebras*, Alg. i Anal. 1 (1989) 178

[12] V. G. Drinfeld, *Quantum Groups*, ICM MSRI, Berkeley (1986) 798-820

[13] M. Jimbo *A q-Difference Analogue of* U(g) *of the Yang-Baxter Equation*, Lett. Math. Phys. 10 (1985) 63-69

[14] N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky, *Quantum R-matrices and Factorization Problems*, JGP. Vol. 5, nr. 4 (1988) 534-550